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Elastic buckling of I-beams under linear moment gradient

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Abstract

This paper investigates the elastic lateral–torsional buckling of I-beams under linear moment gradient that very precisely incorporates the effects of moment gradient and various end restraints. The elastic critical buckling moments are obtained independently by using: (1) the Bubnov–Galerkin method and (2) the finite element method. The present formula of the moment gradient correction factor cannot satisfactorily predict the buckling capacities of doubly symmetric and monosymmetric I-beams with various end restraints. We propose alternative equations for evaluating the moment gradient correction factor, considering end restraint conditions.

Keywords: Elastic lateral–torsional buckling; Moment gradient correction factor; End restraints

1. Introduction

Lateral torsional buckling is a limit state that may often be a controlling factor in steel beam designs. Design specifications usually provide buckling solutions derived for uniform moment loading condition and account for variable moment along the unbraced length with a moment gradient correction factor C_b applied to these solutions.

For a simply supported doubly symmetric I-beam which is prevented from lateral deflection and twisting but free to rotate laterally and warp, the elastic critical uniform moment, M_{ocr}, can be expressed as follows (Trahair, 1977):

$$M_{ocr} = \frac{\pi}{L}\sqrt{EI_yGK_T}\sqrt{1 + W^2} \tag{1}$$

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in which

$$W = \frac{\pi}{L}\sqrt{\frac{EI_w}{GK_T}} \tag{2}$$

EI_y = the minor axis flexural rigidity; GK_T = the St-Venant torsional rigidity; EI_w = warping rigidity; L = the unbraced length of the beam; and W = torsional slenderness parameter.

If the end moments are unequal, the critical moments cannot be predicted by Eq. (1) so an adequate evaluation is required. Most design specifications provide a solution for lateral–torsional buckling of a doubly symmetric I-beam subjected to unequal end moments as follows:

$$M_{cr} = C_b M_{ocr} \tag{3}$$

where M_{ocr} is obtained from Eq. (1), and C_b is a moment gradient correction factor that accounts for the increased resistance to lateral–torsional buckling when the applied loading does not produce constant, or uniform, moment over the entire unbraced length of beams.

The equation for the C_b factor that was used in the 1st edition of AISC Specifications (1986) is

$$C_b = 1.75 + 1.05 \cdot r + 0.3 \cdot r^2 \leqslant 2.3, \quad r = \frac{M_S}{M_L} \tag{4}$$

where r represents an end moment ratio, M_L is the larger end moment, and M_S is the smaller end moment. The end moment ratio, r, is taken as positive for moment causing reverse-curvature bending and negative for single-curvature bending as shown in Fig. 1. Eq. (4) was presented by Salvadori (1955).

Kirby and Nethercot (1979) presented an alternative equation for C_b, which is applicable for any shape of moment diagrams. The equation is

$$C_b = \frac{12.5M_{max}}{2.5M_{max} + 3M_A + 4M_B + 3M_C} \tag{5}$$

where M_{max} is the absolute maximum moment along L, and M_A, M_B, and M_C are the absolute moments at the quarter, the center, and the three-quarter point, respectively.

The 3rd edition of the American Institute of Steel Construction (AISC) *load and resistance factor design* (LRFD) specifications (2001) has incorporated Eq. (5) for C_b. The American Association of State Highway and Transportation Officials (AASHTO) LRFD specifications (1994) has included both Eq. (4) and (5) for C_b.

The SSRC *Guide* (Galambos, 1998) provides a solution for a lateral–torsional buckling moment, M_{cr}^m, of the monosymmetric I-beams subjected to unequal end moments as follows:

$$M_{cr}^m = C_b M_{ocr}^m \tag{6}$$

For a simply supported monosymmetric I-beam under uniform moment, elastic critical uniform moment, M_{ocr}^m, can be expressed as (Kitipornchai and Trahair, 1980; Kitipornchai et al., 1986)

$$M_{ocr}^m = \frac{\pi}{L}\sqrt{EI_y GK_T}\left[\sqrt{1 + W^2 + \frac{1}{4}Q^2} + \frac{1}{2}Q\right] \tag{7}$$

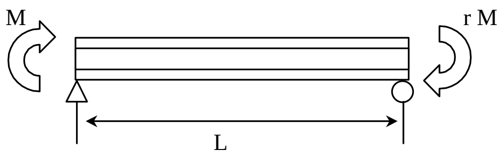


Fig. 1. Simply supported I-beams under linear moment gradient.

in which

$$Q = \frac{\pi}{L} \cdot \beta_x \cdot \sqrt{\frac{EI_y}{GK_T}} \tag{8}$$

The term β_x is the monosymmetric parameter defined as

$$\beta_x = \frac{1}{I_x}\left(\int_A x^2 y \, dA + \int_A y^3 \, dA\right) - 2y_0 \tag{9}$$

in which x and y are coordinates in a cross section based on the centroidal principle system having the x-axis as the major axis, and I_x is the 2nd moment of inertia about the major axis, and y_0 is the coordinate of the shear center.

The finite element program developed by Lim et al. (2002) is used to investigate the accuracy of the present design formulae, Eqs. (4) and (5). The FE model is built with two-node beam elements including the warping degree of freedom. The C_b factors for a simply supported doubly symmetric I-beam computed using Eqs. (4) and (5) are shown in Fig. 2 and Table 1. The C_b factors obtained from a finite element analysis (Lim et al., 2002) are also shown in Fig. 2 and Table 1 for comparison.

The nondimensional buckling moment for a simply supported monosymmetric I-beam computed using Eqs. (4) and (5) is shown in Fig. 3 and Table 2. The nondimensional buckling moment obtained from a finite element analysis is also shown in Fig. 3 and Table 2 for comparison. From these results, we find that the present design formulae, Eqs. (4) and (5), to account for the effects of the moment gradient may be either highly unsafe or overly conservative.

In Fig. 3 and Table 2, the degree of monosymmetry, ρ, is given by (Kitipornchai and Trahair, 1980)

$$\rho = \frac{I_{yC}}{I_{yC} + I_{yT}} \tag{10}$$

in which I_{yC}, I_{yT} are the section minor axis second moments of area of the compression and tension flanges, respectively. Nondimensional buckling moment, γ_c, is defined as

$$\gamma_c = \frac{M_{cr}^{m} \cdot L}{\sqrt{EI_y \cdot GK_T}} \tag{11}$$

Since the lateral–torsional buckling implies three kinds of deformation (twisting, lateral bending, and warping), end restraints have a pronounced effect on the elastic lateral–torsional buckling strength of

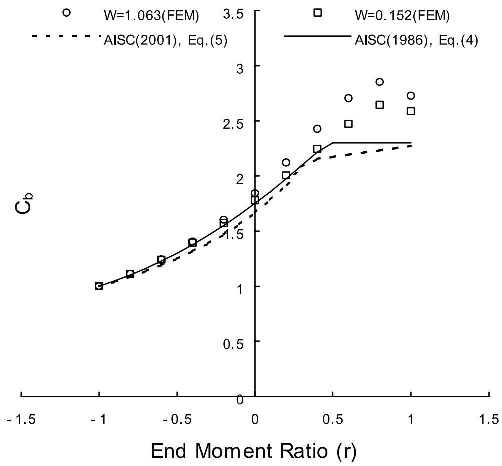


Fig. 2. C_b factors for a simply supported doubly symmetric I-beam under moment gradient.

Table 1
C_b factors for a doubly symmetric I-beam ($W = 1.063$) under moment gradient

End moment ratio (r)	C_b (FEM)	AISC (2001), Eq. (5)		AISC (1986), Eq. (4)	
		C_b	Error (%)	C_b	Error (%)
−1	1	1	0	1	0
−0.8	1.110	1.087	−2.04	1.102	−0.68
−0.6	1.243	1.190	−4.21	1.228	−1.19
−0.4	1.405	1.316	−6.34	1.378	−1.92
−0.2	1.603	1.471	−8.26	1.552	−3.18
0	1.842	1.667	−9.53	1.750	−5.01
0.2	2.121	1.923	−9.33	1.972	−7.02
0.4	2.425	2.155	−11.13	2.218	−8.54
0.6	2.705	2.193	−18.92	2.300	−14.96
0.8	2.852	2.232	−21.73	2.300	−19.35
1	2.726	2.273	−16.62	2.300	−15.62

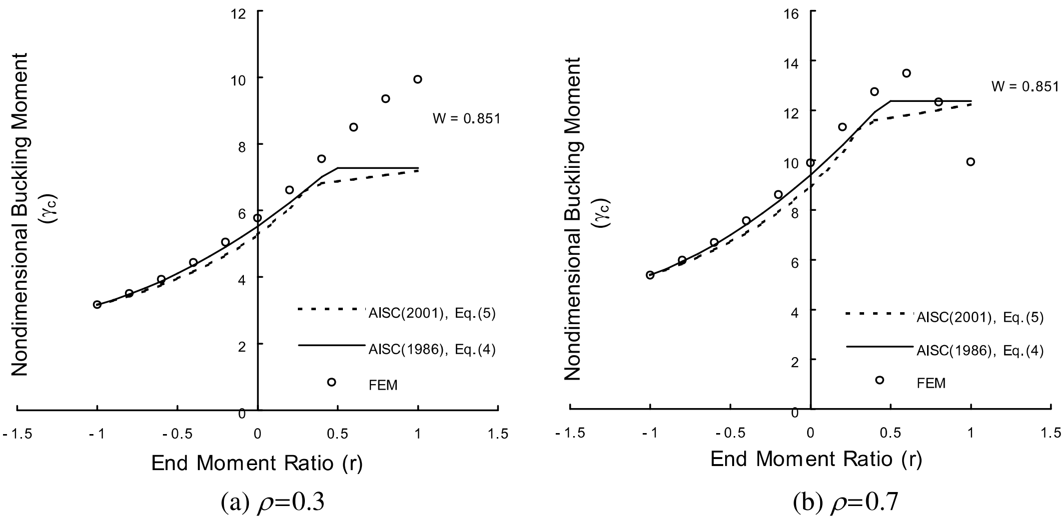


Fig. 3. Nondimensional buckling moment for a simply supported monosymmetric I-beam under moment gradient.

Table 2
Nondimensional buckling moment for a monosymmetric I-beam ($W = 0.851$, $\rho = 0.7$) under moment gradient

End moment ratio (r)	C_b (FEM)	AISC (2001), Eq. (5)		AISC (1986), Eq. (4)	
		C_b	Error (%)	C_b	Error (%)
−1	5.380	5.380	0	5.380	0
−0.8	5.971	5.848	−2.06	5.929	−0.71
−0.6	6.687	6.405	−4.22	6.607	−1.20
−0.4	7.555	7.079	−6.29	7.414	−1.86
−0.2	8.614	7.912	−8.15	8.350	−3.06
0	9.883	8.967	−9.27	9.415	−4.74
0.2	11.325	10.347	−8.64	10.610	−6.31
0.4	12.742	11.595	−9.00	11.933	−6.34
0.6	13.487	11.799	−12.52	12.374	−8.25
0.8	12.325	12.009	−2.56	12.374	0.40
1	9.958	12.228	23.17	12.374	24.65

beams. This paper presents a more accurate solution for the elastic lateral–torsional buckling of I-beams under a linear moment gradient. This solution accurately incorporates the effects of the moment gradient and various end restraints. The elastic critical buckling moments are obtained independently by using: (1) the Bubnov–Galerkin method and (2) the finite element method. These results are then compared with those furnished by the present design formulae.

2. Approximate buckling formula

Consider the monosymmetric I-beam of span L as shown in Fig. 1. The ends of the beam are free to rotate about the major principal axis, but are restrained against twisting about the longitudinal axis. The Bubnov–Galerkin method is applied to find an approximate solution of the following differential equation governing lateral–torsional buckling of beams (Vlasov, 1960).

$$EI_y \cdot u^{IV} + L^2 \cdot (M_x \cdot \theta)'' = 0 \tag{12}$$

$$EI_\omega \cdot \theta^{IV} - L^2 \cdot GK_T \cdot \theta'' + L^2 \cdot M_x \cdot u'' - L^2 \cdot \beta_x \cdot (M_x \cdot \theta')' = 0 \tag{13}$$

where u and θ denote the lateral deflection and twisting, respectively, M_x is the bending moment about the major axis and the prime denotes differentiation with respect to dimensionless variable ξ ($\xi = z/L$, $0 \leqslant z \leqslant L$).

The buckled shape of beams subjected to the linear moment gradient could be reasonably approximated by the two terms for the lateral deflection, u, and one term for the angle of twist, θ.

$$u = A_1 \cdot \chi_1 + A_2 \cdot \chi_2 \tag{14a}$$

$$\theta = B_1 \cdot \varphi_1 \tag{14b}$$

where the functions χ_1, χ_2 and φ_1 are so chosen that the buckled state of beams should satisfy the given boundary conditions as shown in Table 3; $A1$, $A2$, and $B1$ are unknown coefficients. A fourth degree polynomial for the functions χ_1 and φ_1 and a fifth degree polynomial for the function χ_2 are chosen for simplicity of mathematical treatment.

The bending moment at any point along the span can be expressed as

$$M_x = M \cdot \{1 - (1 + r) \cdot \xi\} \tag{15}$$

in which M is the left-hand end moment as shown in Fig. 1.

2.1. Doubly symmetric I-beam

Using the Bubnov–Galerkin approach and approximate functions for u and θ, the elastic buckling moment, M_{cr}, can be calculated by

Table 3
End restraint conditions

End restraints	BC-1 simply supported	BC-2 fixed end	BC-3 warping prevented
Lateral displ.	Prevented	Prevented	Prevented
Lateral bending	Free	Prevented	Free
Twisting	Prevented	Prevented	Prevented
Warping	Free	Prevented	Prevented

$$M_{\mathrm{cr}} = \alpha_1 \cdot \alpha_2 \cdot M_{\mathrm{ocr}} \tag{16}$$

in which α_1 and α_2 are the coefficients that depend on the linear moment gradient and the end restraint condition, respectively.

$$\alpha_1 = \frac{2}{\sqrt{(1-r)^2 + R1 \cdot (1+r)^2}} \tag{17}$$

where $R1$ is the coefficient that depends on the interaction between the moment gradient and the end restraint condition. For BC-1, $R1 = 0.132$; for BC-2, $R1 = 0.156$; and for BC-3, $R1 = 0.087$. These $R1$ values can be used in Eq. (17). In the case of the uniform moment condition ($r = -1$), $\alpha_1 = 1$. For BC-1, Eq. (17) provides an identical result to that of Kitipornchai et al. (1986) using the finite integral method.

α_2 can be calculated by

$$\alpha_2 = \frac{1}{K_u \cdot K_\theta \cdot \sqrt{R3}} \cdot \frac{\sqrt{K_\theta^2 + W^2}}{\sqrt{1 + W^2}} \tag{18}$$

where K_u and K_θ are the effective length factors for the lateral and warping restraint, respectively. For BC-1, $K_u = 1$, $K_\theta = 1$; for BC-2, $K_u = 0.5$, $K_\theta = 0.5$; and for BC-3, $K_u = 1$, $K_\theta = 0.5$. These effective length factors can be used in Eq. (18). In Eq. (18), $R3$ is the coefficient that depends on the difference between the lateral and warping restraint conditions. For BC-1 and BC-2, $R3 = 1$; for BC-3, $R3 = 0.8$.

2.2. Monosymmetric I-beam

The nondimensional elastic buckling moment, γ_c, for the monosymmetric I-beam, can be expressed as follows:

$$\gamma_c = \pi \cdot \alpha_1 \cdot \frac{1}{K_u \cdot \sqrt{R3}} \cdot \left\{ \frac{1}{2} \cdot G1 \cdot \alpha_1 \cdot \frac{Q}{K_u \cdot \sqrt{R3}} + \sqrt{\frac{1}{4} \cdot \left(G1 \cdot \alpha_1 \cdot \frac{Q}{K_u \cdot \sqrt{R3}} \right)^2 + \left(1 + \frac{W^2}{K_\theta^2} \right)} \right\} \tag{19}$$

in which $G1 = (1-r)/2$. For BC-1, Eq. (19) provides an identical result to that of Kitipornchai et al. (1986) using the finite integral method.

3. Finite element method

More accurate solutions for the lateral–torsional buckling of I-beams are obtained from a finite element analysis (Lim et al., 2002) for three end restraint cases over the complete range of the linear moment gradient ($-1 \leqslant r \leqslant 1$). In this finite element program, the axial displacement and transverse displacements are interpolated by the linear and the cubic hermitian shape functions, respectively, and the torsional displacement is interpolated by the homogeneous solution of the following differential equation.

$$EI_{\mathrm{w}} \cdot \theta^{IV} - GK_{\mathrm{T}} \cdot \theta'' = 0 \tag{20}$$

For the eigenvalue extraction, the subspace iteration method (Bathe, 1996) is used.

I-beams are modeled by using the beam element including warping degree of freedom. This beam element has two nodes per element and seven nodal degrees of freedom. The principal generalized coordinate and two-reference line system is adopted; i.e., the centroidal axis for axial and bending action; and the line of shear center for shear, twisting, and warping action.

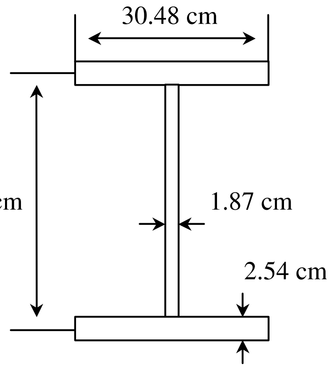


Fig. 4. Basic section considered in finite-element analysis.

A doubly symmetric I-shaped cross section ($\rho = 0.5$) is shown in Fig. 4. The size of the flanges is changed to vary the degree of monosymmetry, Eq. (10), of the cross section. One of the flanges is fixed at 30.48 by 2.54 cm while the size of the other flange is varied. The web thickness is kept at 1.87 cm and the distance between flange centroid is 150 cm for all the sections considered. The degree of monosymmetry, ρ, is varied from 0 to 1. In order to obtain the various torsional slenderness parameters, Eq. (2), various span lengths are used in the finite element analysis.

3.1. Influence of end restraints under uniform moment

In this section the variation of end restraint factors (K_u, K_θ, $R3$) is examined for beams subjected to uniform moment. Fig. 5 is a graph of α_2, Eq. (18), versus W for doubly symmetric I-beams. Results from the FEM are also shown in Fig. 5.

Fig. 6 is a graph of γ_c, Eq. (19) with $\alpha_1 = G1 = 1$, versus W for monosymmetric I-beams. Results from the FEM are also shown in Fig. 6.

Eq. (18) for doubly symmetric sections and Eq. (19) for monosymmetric sections under a uniform moment predict buckling moments that are in very good agreement with the FEM results, except for the

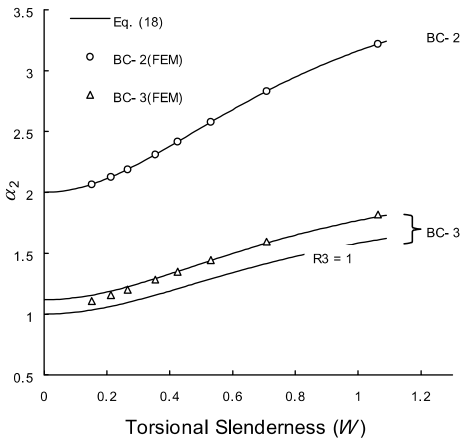


Fig. 5. α_2 vs. W for doubly symmetric I-beams.

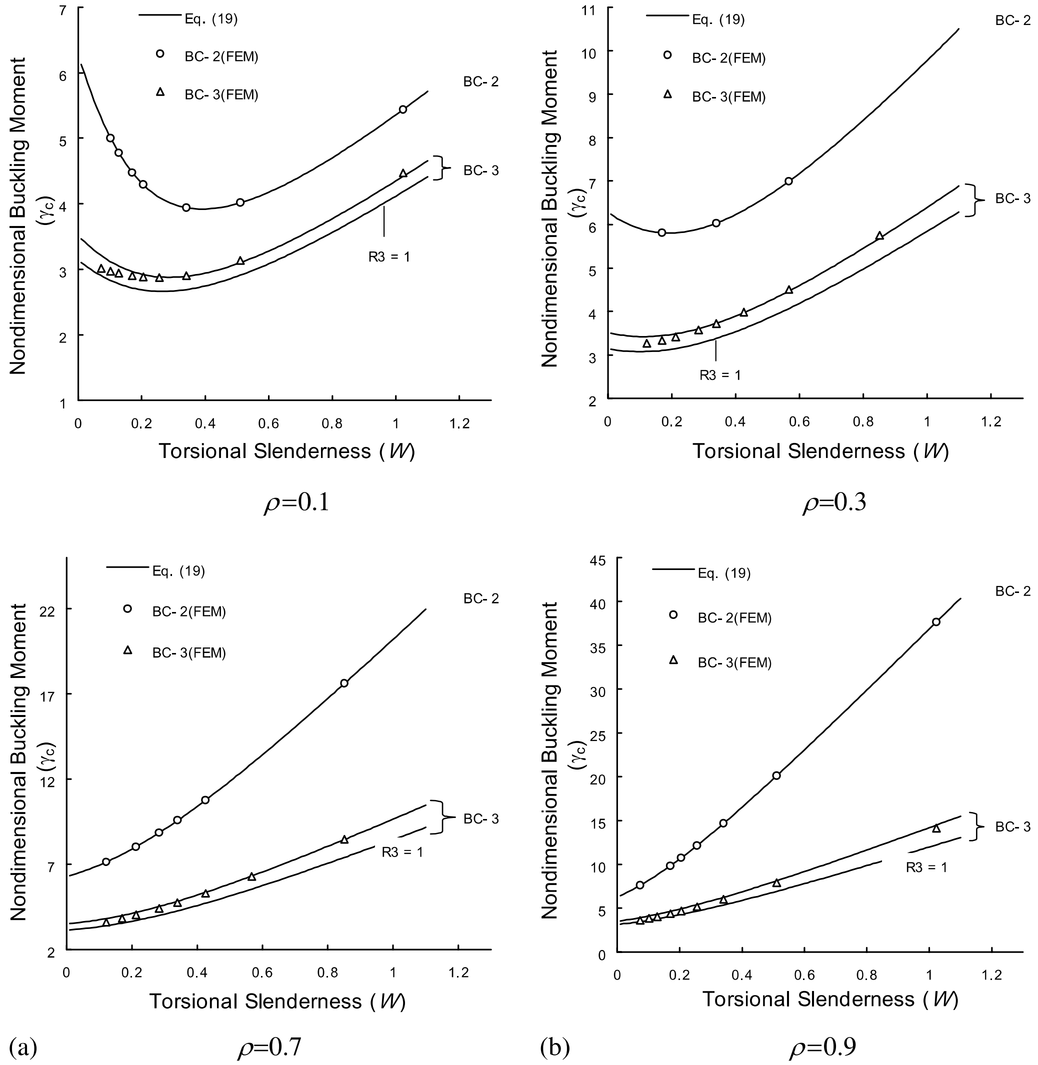


Fig. 6. γ_c vs. W for monosymmetric I-beams.

case of BC-3 with the small value of the torsional slenderness, W. For BC-3, if $R3$ in Eqs. (18) and (19) is 1, then in no case is the error in buckling capacity greater than about 10%, as shown in Figs. 5 and 6.

3.2. Influence of linear moment gradient under various end restraint conditions

3.2.1. Beams with ends simply supported; BC-1

For the range of the degree of monosymmetry from 0.3 to 0.7, a more accurate solution for α_1 may be obtained from finite element results as follows:

$$\alpha_1 = \frac{2}{\sqrt{(1-r)^2 + 0.16 \cdot (1+r)^2}} \tag{21}$$

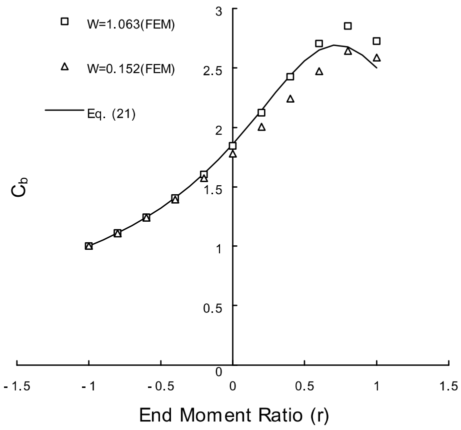
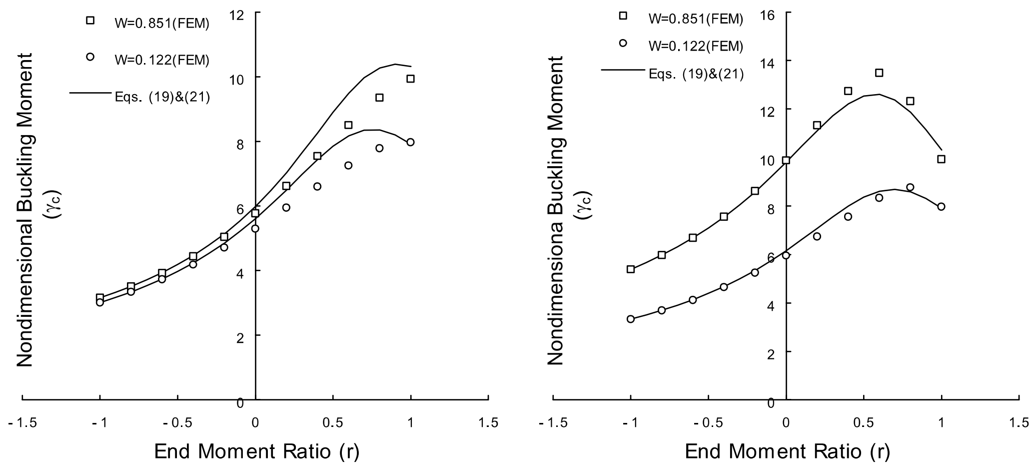


Fig. 7. C_b vs. end moment ratio (r) for doubly symmetric I-beams: BC-1.

The C_b factors for a simply supported doubly symmetric I-beam are computed using $C_b = \alpha_1 \times \alpha_2$ ($\alpha_2 = 1$). The C_b factors using Eq. (21) are shown in Fig. 7. Also, the C_b factors obtained from the finite element analysis are shown in Fig. 7. If instead of Eq. (17), Eq. (21) is used, then no case will show error greater than about 9%. However, as shown in Fig. 2 and Table 1, for AISC (2001) and AISC (1986), the maximum errors do increase to 22% and 19%, respectively.

Fig. 8 shows the results given by the improved approximate formulae, Eqs. (19) and (21), against those obtained by the finite element analysis, for monosymmetric I-beams with $W = 0.851$ and 0.122, respectively. Eqs. (19) and (21) yield values of buckling capacity that are in error by no more than 10%. However, when AISC (1986, 2001) is applied to calculate the buckling capacity, Fig. 3 and Table 2 show errors approaching 25%.



(a) ρ=0.3 (b) ρ=0.7

Fig. 8. γ_c vs. end moment ratio (r) for monosymmetric I-beams: BC-1.

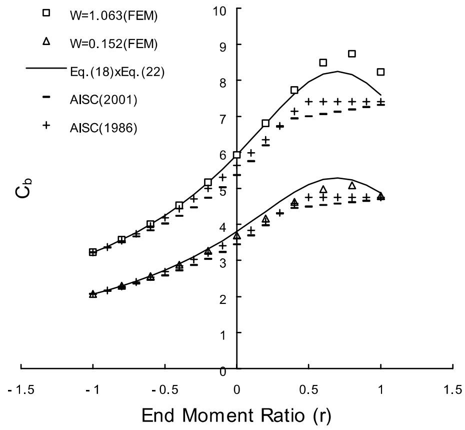


Fig. 9. C_b vs. end moment ratio (r) for doubly symmetric I-beams: BC-2.

3.2.2. Beams with ends completely fixed; BC-2

For the range of the degree of monosymmetry from 0.3 to 0.7, a more accurate solution for α_1 may be obtained from finite element results as follows:

$$\alpha_1 = \frac{2}{\sqrt{(1-r)^2 + 0.18 \cdot (1+r)^2}} \tag{22}$$

Fig. 9 shows the C_b factors given by Eq. (22) × Eq. (18) against those obtained by the finite element analysis, for doubly symmetric I-beams with $W = 1.063$ and 0.152, respectively. The C_b factors (AISC, 1986 = Eq. (4) × Eq. (18) and AISC, 2001 = Eq. (5) × Eq. (18)) of the AISC specifications are also shown in Fig. 9. Eq. (22) gives results that are in error by no more than 8%. Results from Eq. (5) of AISC (2001) and Eq. (4) of AISC (1986) give an error of 18% and 15%, respectively.

Fig. 10 shows the results given by the improved approximate formulae, Eqs. (19) and (22), against those obtained by the finite element analysis, for monosymmetric I-beams with $W = 0.851$ and 0.122, respectively. The nondimensional buckling moments of the SSRC *Guide* (Galambos, 1998) are also shown in Fig. 10. The SSRC *Guide* (Galambos, 1998) provides a solution for nondimensional buckling moment, γ_c, of the monosymmetric I-beams with various end restraint conditions as follows:

$$\gamma_c = \pi \cdot C_b \cdot \frac{1}{K_u} \cdot \left\{ \frac{1}{2} \cdot \frac{Q}{K_u} + \sqrt{\frac{1}{4} \cdot \left(\frac{Q}{K_u}\right)^2 + \left(1 + \frac{W^2}{K_\theta^2}\right)} \right\} \tag{23}$$

in which C_b factor can be calculated from Eq. (4) of the AISC specifications (1986) and Eq. (5) of the AISC specifications (2001).

It can be seen that Eqs. (19) and (22) provide results that are in error by no more than 10%. Eq. (5) of AISC (2001) and Eq. (4) of AISC (1986) give results with errors that are within about 40% and 42% with respect to the finite element results, respectively.

3.2.3. Beams with ends restrained against warping; BC-3

According to results in section 3.1 and Eq. (23) in the SSRC *Guide* (Galambos, 1998), it is recommended that the coefficient $R3$ in Eqs. (18) and (19) be 1. If $R3 = 1$ and degree of monosymmetry ranges from 0.3 to 0.7, then a more accurate solution for α_1 can be obtained from finite element results as follows:

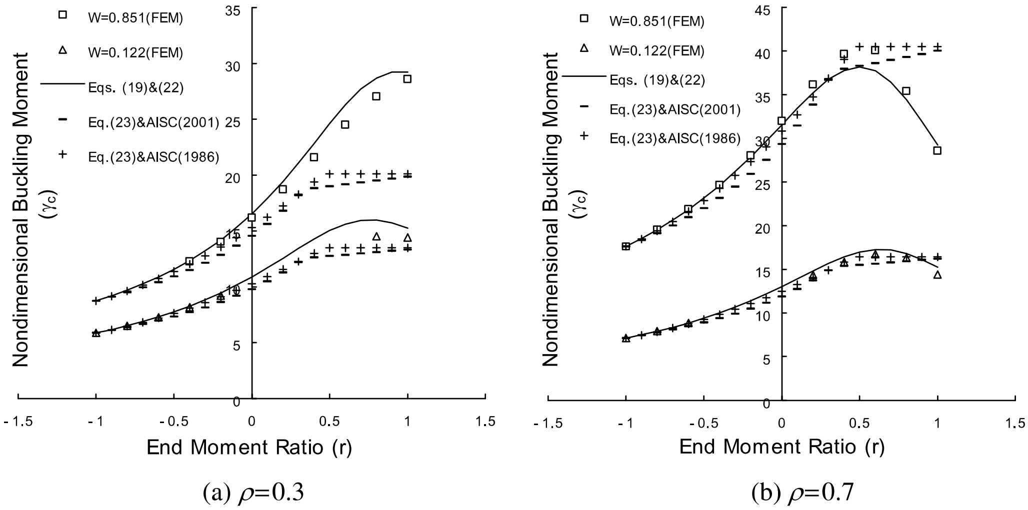


Fig. 10. γ_c vs. end moment ratio (r) for monosymmetric I-beams: BC-2.

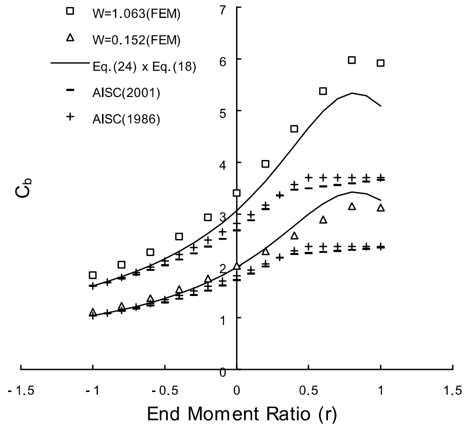
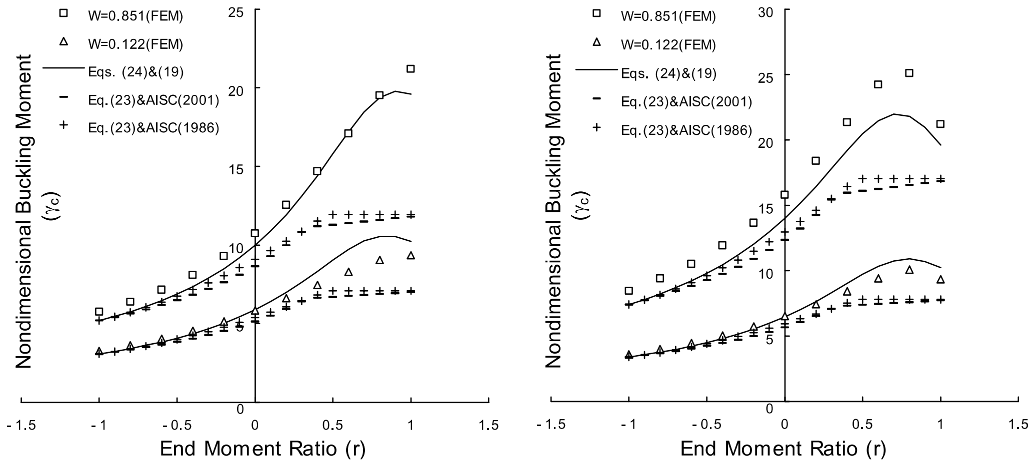


Fig. 11. C_b vs. end moment ratio (r) for doubly symmetric I-beams: BC-3.

$$\alpha_1 = \frac{2}{\sqrt{(1-r)^2 + 0.1 \cdot (1+r)^2}} \tag{24}$$

Fig. 11 shows the C_b factors given by Eq. (24) $\times$ Eq. (18) against those obtained by the finite element analysis, for doubly symmetric I-beams with $W = 1.063$ and 0.152, respectively. The C_b factors of AISC specifications (1986, 2001) are also shown in Fig. 11. It can be seen that Eq. (24) gives results that are in error by no more than about 14%. Eq. (5) of AISC (2001) and Eq. (4) of AISC (1986) give results with errors that are within about 38% with respect to the finite element results.

Fig. 12 shows the results given by the improved approximate formulae, Eqs. (24) and (19), against those obtained by the finite element analysis, for monosymmetric I-beams with $W = 0.851$ and 0.122, respectively. The nondimensional buckling moments of the SSRC *Guide* (Galambos, 1998) are also shown in Fig. 12. It can be seen that Eqs. (19) and (24) give results with errors that are within about 13%. Eq. (5) of AISC



(a) ρ=0.3 (b) ρ=0.7

Fig. 12. γ_c vs. end moment ratio (r) for monosymmetric I-beams: BC-3.

(2001) and Eq. (4) of AISC (1986) give results with errors that are within about 44% with respect to the finite element results.

4. Conclusions

This paper investigates the elastic lateral–torsional buckling of I-beams under linear moment gradient that very precisely incorporates the effects of moment gradients and various end restraints. The elastic critical buckling moments are obtained independently by using: (1) the Bubnov–Galerkin method and (2) the finite element method. For the Bubnov–Galerkin method, a polynomial series is assumed to represent the buckled lateral displacement, u, and the angle of twist, θ. More accurate solutions are obtained from a finite element analysis for three end restraint cases over the complete range of the linear moment gradient.

The application of the present design formula for the moment modification factor and elastic buckling moment is either highly unsafe or overly conservative according to the degree of monosymmetry and end restraint conditions. For the range of ρ from 0.3 to 0.7, the present design formulae (AISC, 1986, 2001; SSRC *Guide*, 1998) give results between −44% and +42% errors with respect to the finite element results. An alternative approximate buckling moment formulae over the range of ρ $(0.3 \leqslant \rho \leqslant 0.7)$ are proposed. The improved formulae yield the buckling capacities of I-beams that are in error by no more than ±14% with respect to the finite element results.

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